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Multi-Portfolio Optimization and Fairness in Allocation of Trades

When trades from separately managed accounts are pooled for execution, the realized market-impact cost can be far greater than the sum of the predicted cost over all accounts. Multi-portfolio optimization is a technique for rebalancing multiple portfolios at the same time, considering their joint effects while adhering to account-specific constraints. The interaction of accounts in a multi-portfolio setting can bias particular accounts if fairness is not considered in the solution methodology. With respect to the trading of multiple accounts, fairness is not well-defined. Definitions vary among portfolio managers often based on their particular investment offering. For this reason, we do not prescribe a single best approach for multi-portfolio optimization. Instead, we discuss the pros and cons of two approaches that each has foundations in economic theory, the Cournot-Nash equilibrium and the collusive solution. We present a unified framework capable of solving either problem.

Multi-Portfolio Optimization and Fairness in Allocation of Trades

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1 Introduction

In a typical Institutional Separately Managed Account (SMA) framework, client portfolios that follow a similar strategy are individually optimized and the resulting trades are *pooled* together for execution. However, this practical trading consideration is rarely incorporated in the individual optimizations of the accounts that participate in the pooled trading. As a result, the expected trading cost of each account is severely underestimated, resulting in a set of trades that is far from optimal.

Modern portfolio rebalancing techniques that rely on optimization use trading costs as an important component of the objective function in the optimization process. Typical models of trading costs include a market impact function that is an increasing nonlinear convex function of the value of each asset being traded. As a result, the estimated market-impact cost computed when each account is rebalanced independently underestimates the total market impact cost that will be incurred when trading the pool of accounts together. That is, the actual market-impact cost of trading multiple accounts is typically much greater than the sum of the estimated market-impact costs of trading each account separately. Therefore, any approach that ignores the aggregate market impact cost is likely to lead to excessive trading and a potential reduction in realized returns for some of the accounts that are being optimized. Because some accounts may be affected by trading cost underestimation more than others, ignoring the aggregate market impact may also result in unintended biases, or *unfair trading*.

Multi-portfolio optimization provides a strategic platform for scaling modern portfolio construction techniques from the individual account level, at which they are originally specified, to the level of a pooled, simultaneous rebalancing. It allows many individual client portfolios with individualized portfolio construction strategies to be optimized simultaneously while considering aggregate market-impact costs or trade-related constraints that better represent realistic trading. Considering the aggregate effect of market-impact costs of the combined trades and properly accounting for them during the rebalancing process ensures

that the resulting portfolios are optimal under the conditions in which they are actually managed and traded.

To date, the aggregate effect of trading several accounts in a pool has often been ignored when the accounts are optimized independently. Although some practitioners may consider the aggregate effects of joint trading, they typically do so only in an ad-hoc, or heuristic, process. One technique frequently used is to aggregate similar accounts, rebalance them together as a single representative account, and then allocate trades to the individual accounts on a pro-rata basis. Even if the trades could be allocated to each of the accounts while adhering to the constraints imposed in each account, this procedure can be non-optimal and even bias certain accounts. Another common procedure is to spread the rebalancing of accounts over the rebalancing cycle in order to reduce the market-impact of one account on another. However, this procedure can also produce sub-optimal solutions, increase account dispersion, and potentially bias certain accounts, thus resulting in unfair trading practices. A more advanced heuristic procedure is to optimize an aggregate multi-portfolio optimization problem using an iterative process where each account is optimized independently, trades are accumulated across all accounts, and constraints and/or objective functions for each individual account are then adjusted to better represent the true cost of aggregate trading. If not based on sound optimization decomposition techniques, such proposed algorithms may not converge and thus be non-optimal and potentially biased.

While striving for optimality should be an important consideration for all algorithms attempting to solve the multi-portfolio optimization problem, it is perhaps even more important to obtain an unbiased solution. For an algorithm to achieve an unbiased solution, it needs to be able to guarantee that the trades will be allocated fairly across accounts, or in other words, that individual investors acting independently, observing the cost of trading of the other accounts, would have acted in a similar manner. It is the fiduciary responsibility of the managers of SMAs to ensure that each account is unbiased. While there is no universally accepted definition of bias with respect to trading, we believe that no account should benefit to the detriment of another.

We do not argue which methods are “fair”. Multi-portfolio optimization has roots in game theory and microeconomics. In game theory, the fairness of a game depends upon the threat of leaving or not playing the game. For an institutional SMA manager, the threat is a client leaving. Do the managers at the institution act independently yet have access to the same information? If so, then the threat to a manager may be different than if the managers within the institution work together to the benefit of all clients.

To the best of our knowledge, O’Cinneide et al. (2006) were first to analyze the simultaneous rebalancing of multiple accounts into a multi-portfolio optimization problem, while considering the notion of fairness or bias. They proposed a multi-portfolio optimization problem where the objective function is the sum of the objective functions of each individual account. In microeconomic theory of oligopolies, this solution is referred to as the *collusive* solution (see Varian, 1984; Henderson and Quandt, 1980). O’Cinneide et al. (2006) claimed that this collusive solution maximizes total welfare over all the accounts, is Pareto optimal, and is fair. They argue the approach is fair because the solution obtained is the same that would have been obtained if each account was competing in an open market for liquidity.

While the solution does indeed maximize the total welfare across all participants, we will later show that it is possible that certain accounts may be better off acting alone than participating in the collusive solution and therefore would be biased if forced to participate.

An alternative methodology for determining the optimal portfolios in a multi-portfolio context is what is known in the literature as the Cournot-Nash equilibrium. Like the collusive solution, this approach originates in the study of the behavior of oligopolies in the microeconomic literature. However, rather than colluding to maximize total welfare, each participant optimizes its own objective assuming the trade decisions of all accounts that participate in the pooled trading have been made and are fixed. In this paper we show how this equilibrium solution can be obtained from the solution to a single multi-portfolio optimization problem. However, the Cournot-Nash solution is not necessarily Pareto optimal which means that each account can potentially do better than its equilibrium solution.

It is not necessarily the purpose of this paper to argue for either the collusive approach or the Cournot-Nash equilibrium over the other. Instead, we show the pros and cons of both approaches and propose a single multi-portfolio rebalancing framework that encompasses either approach. The multi-portfolio optimization approach outlined in this paper allows institutional managers to rebalance multiple accounts simultaneously while ensuring that each account is optimized according to either the collusive solution or the Cournot-Nash equilibrium solution, and adheres to all account-specific constraints as well as any additional constraints that span across multiple accounts.

The remainder of this paper is organized as follows. In Section 2, we describe three different approaches to solving the multi-portfolio optimization problem and discuss the properties of the solution to each. We discuss the issue of fairness and show that, under reasonable behavioral assumptions, no client can benefit by taking any alternative action to that prescribed by the multi-portfolio optimal solution that finds the Cournot-Nash equilibrium. In Section 3, we outline our proposed methodology and describe the flexibility of the modeling framework.

2 Multi-Portfolio Optimization Techniques and Their Properties

When large positions in an equity asset are bought or sold, the price of the security can be affected as the trades are being executed. Therefore, the average price at which the trade is executed is often worse than expected. This implicit cost of trade execution is known as the *market-impact cost*.

Quantitative managers typically model this market-impact cost of trading as an increasing nonlinear function of the amounts traded. Most often, the function used to predict the market-impact cost is expressed as $t^T c(t)$ where t is a vector of the dollar-value traded for each asset and $c(t)$ is a vector function giving the cost per unit traded for each asset. Some commercial optimizers are not able to handle the non-linearities in market impact models, and rely on piecewise-linear approximations of $t^T c(t)$, but the discussions in this paper apply to these models as well. The vector function $c(t)$ is often independent for each asset and

expressed as a polynomial of the form t_k^p for each asset k where p is a rational number between 0.5 and 1 (see Almgren et al., 2005). The results presented in this paper also hold under more general market-impact functions.

It is not necessarily the case that all accounts following a similar strategy will tend to trade the same assets. In fact, some accounts may sell an asset while other accounts may buy the same asset. We are not aware of any published research on market-impact modeling in this regard. In this paper, we will discuss various ways to account for the cost of trades across accounts such as the netting of buys and sells. Though the management of ERISA accounts prevents the crossing of trades between accounts, we would still expect that the cost of the buys and sells to offset each other to some extent even when they are executed by different brokers. One alternative would be to consider the market-impact of buys and sells independently by simply adding a market-impact function of the buy trades to a market-impact function of the sell trades. The exact function used should depend on how the trades are executed and is likely to be fund, or at least institution, specific.

In this section, we will assume that each account has an all-cash initial position and is not allowed to short assets in order to simplify our discussion of the various high-level approaches to multi-portfolio optimization. Under this assumption, each of the suggested alternatives for aggregating trades are equivalent. Let the variable w_i represent the vector of asset holdings for account i in units of currency and let \mathcal{A} be the set of accounts to be rebalanced. Under the assumptions made, we denote the generic market-impact cost function to be $c(w_1; \dots; w_{|\mathcal{A}|})$. Therefore, the market-impact cost of executing the trades in account i is given as $w_i^T c(w_1; \dots; w_{|\mathcal{A}|})$. In order to simplify the argument, we further assume that $c(w_1; \dots; w_{|\mathcal{A}|})$ is linear and represented as $\Omega \sum_{j \in \mathcal{A}} w_j$, where Ω is a symmetric positive semidefinite matrix.

Now that we have all of the background information, let us consider the different alternatives available for solving the multi-portfolio optimization problem. We will examine the optimality of each and also discuss the properties of the solution to each.

2.1 Optimizing Accounts Independently

First, consider a simple example where the objective function for account i is to maximize its utility (represented by the expected return less portfolio variance) less market-impact costs without any constraints. Then, the optimization problem used to determine the portfolio for account i can be written as follows:

$$\underset{w_i \geq 0}{\text{maximize}} \quad \alpha^T w_i - \rho w_i^T Q w_i - w_i^T \Omega w_i, \quad (1)$$

where α is a vector of expected returns, Q is a covariance matrix of asset returns, and ρ is a risk-aversion parameter. The optimal solution of (1) can be written as

$$\begin{aligned} w_i &= (2\rho Q + 2\Omega)^{-1} (\alpha + \lambda_i), \\ \lambda_{ik} w_{ik} &= 0 \text{ for each asset } k, \\ \lambda_i &\geq 0, w_i \geq 0. \end{aligned} \quad (2)$$

Property	Account 1	Account 2
Size (\$)	100M	10B
Predicted Risk (%)	10.0	10.0
Expected Return (%)	14.54	11.72
Expected Market Impact (%)	0.1821	2.5192
Actual Market Impact (%)	4.4403	2.5618
Expected Objective (%)	14.36	9.20
Actual Objective (%)	10.10	9.16
Change in Objective (%)	-29.7	-0.4

Table 1: Summary Statistics of Independent Account Rebalancing Example

In this problem, each account is acting independently and is unaware of the effect on the market impact cost that the other accounts have. If each account is optimized according to the same strategy, then the actual market-impact cost to account i is $w_i^T \Omega \left(\sum_{j \in \mathcal{A}} w_j \right)$ rather than the $w_i^T \Omega w_i$ that was estimated. The actual market-impact cost to each account is $100(|\mathcal{A}| - 1)\%$ greater than expected, e.g., if the number of accounts is 100, then the actual market-impact cost would have been 9900% greater than expected.

In this scenario, information of the trades being executed is not shared amongst the accounts being rebalanced. It is clear that each account’s solution is suboptimal because of the under-estimation of market impact costs for each account that is being rebalanced. However, this in itself does not mean that any particular account is biased. The question is whether or not any particular account has a consistent advantage over others in such a scenario.

Suppose that there are only two accounts, one with one-hundred million dollars (\$100M) and one with ten billion dollars (\$10B) in total assets, and that both start from cash positions. Furthermore, suppose that our problem is to maximize expected return less market-impact costs subject to being fully invested and a 10% risk constraint.¹ Each account is optimized completely independently of the other. The results of the portfolio rebalancings are summarized in Table 1. In Table 1, the “Expected” market-impact and objective fields calculate the expected market-impact costs using only the trades of the single account while the “Actual” fields calculate the market-impact costs using the combined trade amounts of both accounts. The “Actual Objective” will also be referred to as the *expected net alpha*, having adjusted the expected returns by the expected market-impact costs. Note the large decrease in the objective value for the \$100M account. Because the trades on the large account have a much greater effect on the market-impact cost of the smaller account than vice-versa, the smaller account is consistently being hurt by the approach, thus there is a valid argument that this approach negatively affects the smaller account.

In summary, when each account is optimized independently under the simplistic scenario given here, each account incurs a greater market-impact cost than expected. Furthermore,

¹Note that this problem is slightly different from that modeled in (1)

unintended biases can occur that make some participants particularly worse-off by optimizing them independent of the others as shown in the example above.

2.2 Competitive Equilibrium

As we have seen, treating the market-impact in a single-account optimization as if it is the only account being traded underestimates the true trading cost of rebalancing each account. Instead, the market-impact caused by all accounts being optimized simultaneously should be considered.

Now, assume that the rebalancing problem for each account is “made aware” of the trade amounts to be made amongst all other accounts that will be traded together. Then, we can incorporate the information about these additional trades in the rebalancing problem for each account. Here, let us temporarily assume that the actions of all other accounts are known and fixed. Then, the values of $w_j \geq 0$ are fixed for each account $j \in \mathcal{A} \setminus i$. Under this scenario, the optimization problem for account i can be modeled as follows:

$$\underset{w_i \geq 0}{\text{maximize}} \quad \alpha^\top w_i - \rho w_i^\top Q w_i - w_i^\top \Omega \left(\sum_{j \in \mathcal{A}} w_j \right). \quad (3)$$

The optimal solution to (3) is given by

$$\begin{aligned} w_i &= (2\rho Q + 2\Omega)^{-1} \left(\alpha + u_i - \Omega \sum_{j \in \mathcal{A} \setminus i} w_j \right), \\ u_{ik} w_{ik} &= 0 \text{ for each asset } k, \\ u_i &\geq 0, \quad w_i \geq 0. \end{aligned} \quad (4)$$

We then return to the case where w_j is not known for each $j \in \mathcal{A} \setminus i$. The solution satisfying the optimality conditions given in (4) for each account $i \in \mathcal{A}$ is the competitive equilibrium solution. In microeconomics, this equilibrium is referred to as the Cournot-Nash equilibrium (see Varian, 1984; Henderson and Quandt, 1980). We will later show how this problem is solved. For now, we are only interested in its solution properties. In the Cournot-Nash equilibrium, the actual market-impact cost to each account is exactly what the account expected. Therefore, the Cournot-Nash equilibrium solution is superior to the independent account strategy discussed in Section 2.1.

To illustrate the advantage of the Cournot-Nash solution over optimizing each account independently, consider the same two-account example introduced in Section 2.1. The results comparing the Cournot-Nash equilibrium solution to the individual account optimizations are summarized in Table 2. Note that Account 1 has a greater actual objective in the Cournot-Nash equilibrium solution than it does in the individual account optimization. Account 2 has a greater actual objective as well, though the difference is so small that it does not show up in the number of significant digits represented in the table. Furthermore, notice that the aggregate actual objective across both accounts is greater for the equilibrium solution than for the individual account optimizations.

The Cournot-Nash equilibrium solution is fair to all investors in the sense that no investor will have incentive to deviate from their Cournot-Nash portfolio unilaterally. Since the

Property	Individual Solutions		Equilibrium Solution	
	Account 1	Account 2	Account 1	Account 2
Size (\$)	100M	10B	100M	10B
Predicted Risk (%)	10.0	10.0	10.0	10.0
Expected Return (%)	14.54	11.72	14.39	11.69
Expected Market Impact (%)	0.1821	2.5192	4.1543	2.5329
Actual Market Impact (%)	4.4403	2.5618	4.1543	2.5329
Expected Objective (%)	14.36	9.20	10.24	9.16
Actual Objective (%)	10.10	9.16	10.24	9.16
Aggregate Objective (%)	9.1651		9.1685	

Table 2: Summary Statistics of Cournot-Nash Equilibrium Example

Cournot-Nash equilibrium optimizes the net alpha objective for each individual account, the resulting solution dominates the solution obtained by optimizing each account individually, where the market-impact cost is generally under-estimated. However, the Cournot-Nash solution is not necessarily the ideal solution. It is well-known that the Cournot-Nash solution is not generally Pareto optimal meaning that there exists a solution where at least one account can improve without negatively impacting any other account.

2.3 Collusive Solution

One way to obtain a Pareto optimal solution is to sum the objective functions of the individual account optimization problems defined in (3). This is what we term the collusive approach, where all investors collude to maximize the total welfare over all accounts. This results in a single optimization problem which can be written as follows:

$$\underset{w_i \geq 0 \forall i \in \mathcal{A}}{\text{maximize}} \quad \sum_{j \in \mathcal{A}} \alpha^T w_j - \rho \sum_{j \in \mathcal{A}} w_j^T Q w_j - \left(\sum_{j \in \mathcal{A}} w_j \right)^T \Omega \left(\sum_{j \in \mathcal{A}} w_j \right). \quad (5)$$

The optimal solution to (5) is given by

$$\begin{aligned} w_i &= (2\rho Q + 2\Omega)^{-1} \left(\alpha + v_i - 2\Omega \sum_{j \in \mathcal{A} \setminus i} w_j \right) \quad \forall i \in \mathcal{A}, \\ v_{ik} w_{ik} &= 0 \text{ for each asset } k \text{ and } \forall i \in \mathcal{A}, \\ v_i &\geq 0, \quad w_i \geq 0 \quad \forall i \in \mathcal{A}. \end{aligned} \quad (6)$$

While the solution given by (6) maximizes the total utility, as represented by the sum of the individual utilities less market-impact costs of all accounts, the question is whether or not this is fair to each individual account.

We now consider the same two-account example and compare the collusive solution to the Cournot-Nash equilibrium. A summary of the results is given in Table 3. Note that Account 1 is much better off using the equilibrium solution than the collusive solution. By relying on the collusive solution, Account 1 would be reducing its expected net alpha by 107

Property	Equilibrium Solution		Collusive Solution	
	Account 1	Account 2	Account 1	Account 2
Size (\$)	100M	10B	100M	10B
Predicted Risk (%)	10.0	10.0	10.0	10.0
Expected Return (%)	14.39	11.69	11.69	11.69
Actual Market Impact (%)	4.1543	2.5329	2.5207	2.5207
Actual Objective (%)	10.24	9.16	9.17	9.17
Aggregate Objective (%)	9.1685		9.1732	

Table 3: Summary Statistics of Collusive Example

basis points. However, the aggregate welfare across both accounts is greater in the collusive solution. So, the investors are colluding to increase the expected net alpha of the larger account by approximately one basis point to the detriment of the smaller account. The accounts behave this way for the sake of improving total welfare. In both the individual and Cournot-Nash equilibrium solutions, Account 1 is able to invest in higher alpha, but less liquid, assets to improve its expected net alpha. In the collusive solution presented for the example, the percentage holdings are equivalent for both accounts.

When participating in a collusion, some of the individuals may make sacrifices for the good of others in order to maximize total welfare. Therefore, in order to maintain fairness amongst each participant, the total welfare must be shared appropriately. Even in the event that a fair solution to the collusive problem is found through the appropriate sharing of the combined benefits, we argue that such collusion and sharing of wealth amongst separately managed accounts could not be implemented in practice.

It is interesting to note that the collusive solution can also be obtained by assuming that the objective function for each account is a specific function that is different from the individual account rebalancing and obtaining a Cournot-Nash equilibrium. To see this, consider the following optimization problem:

$$\underset{w_i \geq 0}{\text{maximize}} \quad \alpha^\top w_i - \rho w_i^\top Q w_i - w_i^\top \Omega w_i - 2w_i^\top \Omega \left(\sum_{j \in \mathcal{A} \setminus i} w_j \right). \quad (7)$$

The optimal solution to (7) is given by

$$\begin{aligned} w_i &= (2\rho Q + 2\Omega)^{-1} \left(\alpha + u_i - 2\Omega \sum_{j \in \mathcal{A} \setminus i} w_j \right), \\ u_{ik} w_{ik} &= 0 \text{ for each asset } k, \\ u_i &\geq 0, \quad w_i \geq 0. \end{aligned} \quad (8)$$

If we write (8) for each account, we get the exact same optimality conditions as we did for the collusive solution given by (6).

This analysis can also be used to explain why the net alpha for Account 1 is less in the collusive solution than in the Cournot-Nash solution. In the modified objective function given in (7) used to obtain a Cournot-Nash equilibrium, each account now believes that

it must pay a market-impact cost that would result from its own trades as well as twice those of the other accounts. Because Account 2 is larger, the greater expected return for the relatively illiquid assets that are available for Account 1 no longer appear attractive because the rebalancing problem assumes that Account 2 is actually purchasing twice as much as it really is.

3 Multi-Account Rebalancing Solution

In this section, we describe our general framework for multi-account optimization that encompasses both the collusive and the Cournot-Nash equilibrium approaches.

Recall that variable w_i represents the vector of asset holdings and the parameter h_i represents the vector of initial holdings for account i . Denote the vector of amounts traded of each asset from account i as $t_i = w_i - h_i$. We denote the net amounts traded across all accounts as $t = \sum_{i \in \mathcal{A}} t_i$, where \mathcal{A} is the set of all accounts. Lastly, given a vector of trades, we denote the vectors of buys and sells as t_i^+ and t_i^- , respectively, where $t_i = t_i^+ - t_i^-$.²

Let the objective function of each individual account i be written as $f_i(w_i) - t_i^T c(t)$, where $c(t)$ is a vector function giving the market-impact cost per unit of currency traded and $f_i(w_i)$ represents the expected return, or any more complex utility function containing other terms, such as transaction costs, taxes, risk, etc.

Alternatively, one might use the following objective:

$$f_i(w_i) - (t_i^+)^T c\left(\sum_{j \in \mathcal{A}} t_j^+\right) - (t_i^-)^T c\left(\sum_{j \in \mathcal{A}} t_j^-\right).$$

Note that with this objective function, no account can actually gain from the market impact cost term. On the other hand, one can see that if the account buys some asset, while the net trade across all accounts is a sell, then the term $-t_i^T c(t)$ could make a positive contribution to the objective value.

We assume that the investment restrictions of each account are represented by the vector of inequalities, $g_i(w_i) \geq 0$. Now the optimization problem for each account when the effect of trades of other accounts is considered can be written as follows:

$$\begin{array}{ll}
 \text{maximize} & f_i(w_i) - t_i^T c(t) \\
 \text{subject to} & t_i = w_i - h_i \\
 & t = \sum_{j \in \mathcal{A}} t_j \\
 & g_i(w_i) \geq 0
 \end{array}
 \quad \text{or} \quad
 \begin{array}{ll}
 \text{maximize} & f_i(w_i) - (t_i^+)^T c\left(\sum_{j \in \mathcal{A}} t_j^+\right) \\
 & - (t_i^-)^T c\left(\sum_{j \in \mathcal{A}} t_j^-\right) \\
 \text{subject to} & t_i = w_i - h_i \\
 & g_i(w_i) \geq 0.
 \end{array} \tag{9}$$

In order to correctly optimize all accounts simultaneously, we solve an aggregate optimization problem involving all accounts. For the collusive approach, the aggregate optimization problem adds the objectives together while enforcing the constraints for each account. The

²All variables and parameters represent currency values.

collusive aggregate optimization problem can be written as follows:

$$\begin{aligned}
 & \text{maximize} && \sum_{i \in \mathcal{A}} f_i(w_i) - t^T c(t) \\
 & \text{subject to} && t = \sum_{i \in \mathcal{A}} (w_i - h_i) \\
 & && g_i(w_i) \geq 0 \quad \forall i \in \mathcal{A}. \\
 & && \text{or} \\
 & \text{maximize} && \sum_{i \in \mathcal{A}} f_i(w_i) \\
 & && - \left(\sum_{i \in \mathcal{A}} t_i^+ \right)^T c \left(\sum_{i \in \mathcal{A}} t_i^+ \right) \\
 & && - \left(\sum_{i \in \mathcal{A}} t_i^- \right)^T c \left(\sum_{i \in \mathcal{A}} t_i^- \right) \\
 & \text{subject to} && t_i^+ - t_i^- = w_i - h_i \quad \forall i \in \mathcal{A} \\
 & && t_i^+, t_i^- \geq 0 \quad \forall i \in \mathcal{A} \\
 & && g_i(w_i) \geq 0 \quad \forall i \in \mathcal{A}.
 \end{aligned} \tag{10}$$

How one obtains a Cournot-Nash equilibrium solution depends on the particular market-impact function used. Suppose, for example, that $c(t) = \Omega t$. Then one can show that the Cournot-Nash equilibrium problem can be obtained by solving the following single multi-portfolio optimization problem:

$$\begin{aligned}
 & \text{maximize} && \sum_{i \in \mathcal{A}} f_i(w_i) - \frac{1}{2} \sum_{i \in \mathcal{A}} t_i^T \Omega t_i - \frac{1}{2} t^T \Omega t \\
 & \text{subject to} && t_i = w_i - h_i \quad \forall i \in \mathcal{A}, \\
 & && t = \sum_{i \in \mathcal{A}} t_i \\
 & && g_i(w_i) \geq 0 \quad \forall i \in \mathcal{A}. \\
 & && \text{or} \\
 & \text{maximize} && \sum_{i \in \mathcal{A}} f_i(w_i) - \frac{1}{2} \sum_{i \in \mathcal{A}} (t_i^+)^T \Omega t_i^+ - \frac{1}{2} \left(\sum_{i \in \mathcal{A}} t_i^+ \right)^T \Omega \left(\sum_{i \in \mathcal{A}} t_i^+ \right) \\
 & && - \frac{1}{2} \sum_{i \in \mathcal{A}} (t_i^-)^T \Omega t_i^- - \frac{1}{2} \left(\sum_{i \in \mathcal{A}} t_i^- \right)^T \Omega \left(\sum_{i \in \mathcal{A}} t_i^- \right) \\
 & \text{subject to} && t_i^+ - t_i^- = w_i - h_i \quad \forall i \in \mathcal{A}, \\
 & && t_i^+, t_i^- \geq 0 \quad \forall i \in \mathcal{A} \\
 & && g_i(w_i) \geq 0 \quad \forall i \in \mathcal{A}.
 \end{aligned} \tag{11}$$

Problems (10) and (11) represent “generic” multi-portfolio problems that are representative of practical instances, albeit simple enough to keep mathematical arguments straightforward. In practice, a more complex problem can be solved with the same approach. For example, we might allow for combinatorial constraints, and constraints on holding or trading variables across accounts. In most of these cases, a similar optimization problem could be created having the same desirable properties of optimality and fairness for all. Combinatorial constraints such as thresholds or limits on the number of assets held could also be modeled although some of the desirable properties of the optimal solution may be lost. Similarly, constraints across all accounts could also be added and solved using the same approach. For example, a constraint on the total amount traded in each asset could be imposed.

Problems (10) or (11) can be solved by the same type of direct methods used to solve single period problems. In our experience, such solution methods scale quadratically as a function of the number of accounts on a single processor. A decomposition-based approach

could be used to enhance the performance of the optimizer, and achieve better than quadratic scaling, especially in the presence of a parallel architecture.

While there may not be agreement regarding which variant of the multi-portfolio rebalancing problem should be used (e.g. Collusive versus Cournot-Nash), in this section we showed that these different variants can be handled through a common optimization framework. By using such a generic framework, practitioners have the opportunity to experiment with different options to find the best fit for their investment and trading environment.

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