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Anthony Renshaw, PhD  
Robert A. Stubbs, PhD  
Stefan Schmieta, PhD  
Sebastian Ceria, PhD

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## **Axioma's Alpha Factor Method: Improving Risk Estimation By Reducing Risk Model Portfolio Selection Bias**

**By Anthony A. Renshaw, Robert A. Stubbs, Stefan Schmieta, & Sebastian Ceria  
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### **Abstract**

A new method is described for using factor risk models when estimating portfolio risk and constructing optimal portfolios. This patent-pending method, designated the Alpha Factor Method, adjusts the factor risk model risk estimate of a portfolio suspected of having underestimated risk. This reduces the risk underestimation bias that is unavoidable for optimized portfolios. Empirical backtests indicate that use of the Alpha Factor Method improves the accuracy of risk estimation and leads to superior portfolio construction for optimized portfolios.

### **1. Introduction.**

Selection bias occurs when a subset of an unbiased population is biased due to the method by which the subset is selected. For example, one expects to find more than the average number of gamblers in Las Vegas, actors in Hollywood, and senior citizens in the Sun Belt. In finance, mutual fund analyses frequently suffer from survivorship bias; under-performing funds that have died are not included in the analysis, leading to a disproportionate representation of the surviving, over-performing funds in the set of funds analyzed.

Commercial factor risk model vendors construct risk models using algorithms and data sampling procedures explicitly designed to generate as accurate and as unbiased

a risk model as possible (see for example, Grinold and Kahn (2000) and Litterman (2003)). Institutional investment managers use these risk models to manage risk, construct portfolios, and evaluate manager and fund performance. Of these three tasks, portfolio construction involves selecting a subset of the population of all possible portfolios and is therefore most susceptible to selection bias.

A risk model can be thought of as producing either accurate or inaccurate estimates of either low or high risk forecasts. This leads to the situation shown in Figure 1 below.

Forecasted Risk Level	High	False Negatives	True Negatives
	Low	False Positives	True Positives
		Low	High
		Risk Model Accuracy	

**Figure 1.** Risk model performance decomposed by risk estimation accuracy and the forecasted risk level.

Risk models can result in inaccurate estimates of either Low (Positive) or high (Negative) risk forecasts, or, conversely, accurate estimates of either Low or High risk forecasts. The former will result in False Positives and the False Negative, while the latter will results in True Positives and True Negatives, respectively.

Commercial risk model vendors are interested in moving as many portfolios as possible from the two left boxes to the two right boxes so that the overall accuracy of the risk model is high. These vendors do not differentiate between False Positive and False Negative portfolios.

However, portfolio construction and optimization procedures, whether human or automated, typically attempt to minimize risk for a given level of expected return. Therefore, the subset of portfolios considered only includes the bottom two boxes in Fig. 1, the False Positives and the True Positives. If a portfolio is estimated to have high risk, the portfolio is excluded from further consideration regardless of whether or not that risk estimate is accurate. The top two boxes in Fig. 1 are not relevant to portfolio construction since they entail high risk and are unattractive to investors. The box of primary potential danger for risk model use with portfolio construction is the False Positive box. Portfolio managers and optimization algorithms find low risk portfolios attractive regardless of whether or not the risk estimate is accurate; however, realized tracking error is likely to be significantly higher than predicted tracking error if portfolios lie in the False Positive box.

If the risk model has no risk estimation bias for all possible portfolios (i.e., all four boxes), then the process of selecting low risk portfolios can bias the risk model towards underestimation because underestimated risk portfolios will be disproportionately represented. All other things being equal, an underestimated risk portfolio is more likely to have low risk than an overestimated risk portfolio. In the same way that the set of basketball players is biased towards tall people, the set of low risk or optimized portfolios is biased towards risk underestimation. This bias is an artifact of choosing low risk portfolios and will be true for all risk models, no matter how accurate or robust the risk models may be.

The recognition that risk models may be biased for low risk or optimized portfolios is not new. Muller (1993) studied this effect and estimated that the bias may be as high as 20%. Michaud (1989, 1998) has published several studies demonstrating the fact that optimized portfolios often include False Positives. He calls this effect “error maximization”, which is somewhat misleading because a true error maximizing procedure would only contain False Positive portfolios while optimized portfolios are often True Positives.

Axioma has recently filed a patent for a new method called the Alpha Factor Method which identifies underestimated risk portfolios and compensates for this underestimation. The compensation can either be a correction that makes the estimated risk more accurate or it can simply be a penalty that makes portfolios with underestimated risk less attractive to an optimizer. The primary intellectual insight behind

the method is the recognition that the matrix of asset factor exposures can be used to identify portfolios whose risk is likely to be underestimated. Extensive numerical backtests have been performed that show that the Alpha Factor Method improves the accuracy of traditional risk models and overall optimized portfolio performance.

Although the Alpha Factor Method can be described as the addition of a new factor to a risk model, the method is qualitatively different than previous proposals for improving risk models by adding factors. For example, Miller (2005) proposes improving risk model accuracy by adding statistical factors to a traditional fundamental risk model. The resulting hybrid model contains more factors than the original fundamental risk model, and, not surprisingly, the hybrid model is more accurate in the backtests performed. Although such studies remain important to commercial risk model developers, this approach has been intensively researched for more than two decades and is unlikely to produce more than incremental risk model improvements.

Another area of active research for risk model improvement attempts to capture the uncertainty in the parameters used in the risk model specification. This area of research includes the averaging approach of Michaud (1989, 1998) as well as explicit robust optimization techniques that determine the best, worst-case outcome portfolio given a probabilistic specification of the risk model (Lobo, 2000; Goldfarb and Iyengar, 2003; Ceria and Stubbs, 2005) Although this makes the specification of the risk model more complete, it does not improve the accuracy of the risk model. In reference to Fig. 1, these methods effectively draw a horizontal line across the boxes and guarantee that the portfolio lies above that line. These methods do not attempt to distinguish whether or not any particular portfolio's risk estimate is accurate.

In contrast to prior research, Axioma's Alpha Factor Method explicitly identifies False Positive portfolios and prevents optimizers from using those inaccurate risk assessments during the construction process. To the authors' knowledge, this approach is entirely new. The numerical tests conducted demonstrate that the method adds substantial value to the portfolio construction process for a wide range of investment strategies.

## 2. Mathematical Description of the Alpha Factor Method.

In a factor risk model with  $M$  factors, the asset covariance matrix for  $N$  assets is given by:

$$Q = B\Sigma B^T + \Delta^2 \quad (1)$$

where

$Q$  is an  $N \times N$  covariance matrix,

$B$  is an  $N \times M$  matrix of factor exposures (also called factor loadings),

$\Sigma$  is an  $M \times M$  matrix of factor covariances, and

$\Delta^2$  is a diagonal  $N \times N$  matrix of security specific risk variances.

It is assumed that  $M \ll N$ . It is also assumed that an  $N$ -dimensional column vector of portfolio weights  $w$  is known, where  $w$  represents the fraction of available wealth invested in each asset or security or, for active risk, the difference in weights between a managed portfolio and a benchmark portfolio. The portfolio risk,  $\sigma_p$ , is measured by the volatility of the portfolio, which is the square root of the portfolio variance:

$$\sigma_p = \sqrt{w^T Q w}. \quad (2)$$

It is assumed that the factor risk model is unbiased with respect to under and over estimation of risk across the class of all possible portfolios. Indeed, the algorithms used to construct factor risk models are explicitly designed to minimize this bias. However, when the class of all possible portfolios is narrowed to a family of portfolios with low estimated risk such as optimized portfolios, one may suspect that the risk model has an underestimation bias because portfolios with underestimated risk will be over-represented. The Alpha Factor Method improves risk model use for families of portfolios with low estimated risk by identifying portfolios whose risk is likely to be underestimated

and adjusting the risk estimate for those portfolios to either improve the risk estimate or simply make these portfolios less attractive to a portfolio construction algorithm.

One type of low-risk portfolio that may have underestimated risk is the set of portfolios  $w$  that lie in the null space of  $B^T$ , i.e., portfolios such that  $B^T w = 0$ . The risk of such a portfolio is

$$\sigma_p = \sqrt{w^T B \Sigma B^T w + w^T \Delta^2 w} = \sqrt{w^T \Delta^2 w}. \quad (3)$$

These portfolios only contain specific risk, which is measured by  $\Delta^2$ . No risk is associated with, attributable to, or explained by the factors of the factor risk model. This is the lower bound contribution to risk from the risk model factors. Furthermore, since  $\Delta^2$  is diagonal, the contributions to risk from each asset are uncorrelated.

We believe that risk estimates for portfolios that have exposure to the null space of  $B^T$  are often underestimated. Our rationale is the following. The specific risk associated with  $\Delta^2$  is constructed to be unbiased for all possible portfolios. For portfolios lying in the null space of  $B^T$ , it is also likely, or at least plausible, that the specific risk estimate remains unbiased. If that is true, then the total risk estimate for portfolios with exposure to the null space of  $B^T$  will be unbiased only if the actual factor risk is identically zero. If, in fact, there is some risk attributable to the factors for these portfolios, then the risk model estimated is underestimated. We believe that for most of the portfolios with exposure to the null space of  $B^T$ , the factors explain a small but non-zero part of the total risk. These are the portfolios that are underestimated by the risk model.

Axioma's Alpha Factor Method assumes that risk estimates are underestimated by the extent to which a portfolio has exposure to (i.e., projects into) the null space of  $B^T$  and adjusts the risk estimation accordingly. To do this, the Alpha Factor Method

adds a new, unspecified factor to the original factor risk model which is orthogonal to the factors of the original risk model. A modified factor risk model,  $\tilde{Q}$ , is defined by

$$\tilde{Q} = \begin{bmatrix} B & f \end{bmatrix} \begin{bmatrix} \Sigma & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} B & f \end{bmatrix}^T + \Delta^2, \quad (4)$$

where  $f$  is an  $N$ -dimensional column vector of as yet unspecified factor exposures. The modified risk estimate,  $\tilde{\sigma}_p$ , predicted by  $\tilde{Q}$  is;

$$\tilde{\sigma}_p = \sqrt{w^T \tilde{Q} w} = \sqrt{w^T B \Sigma B^T w + (w^T f)^2 + w^T \Delta^2 w} \quad (5)$$

The modified risk estimate is always greater than or equal to the original estimate since the correction term  $(w^T f)^2$  is always non-negative.

Our goal is to make the magnitude of the risk correction a function of the degree to which the portfolio has exposure to the null space of  $B^T$ . While there are many possible approaches that will do this, we have found the following to be effective: let  $P = I - B(B^T B)^+ B^T$  be the projection matrix into the null space of  $B^T$ , where  $I$  is the identity matrix and  $X^+$  indicates the pseudo-inverse of  $X$ . We define the new factor exposures as  $f = \sigma P w / \|P w\|$  where  $\| \cdot \|$  indicates the Euclidean norm and  $\sigma$  is a fixed, exogenously prescribed constant giving the vector magnitude of  $f$ . The correction term in the modified factor risk model then becomes

$$(w^T f)^2 = \left( \sigma \frac{w^T P w}{\sqrt{w^T P^T P w}} \right)^2 = \sigma^2 w^T P w. \quad (6)$$

One of the advantages of this formulation of the method is that it can be efficiently combined with mean-variance optimization portfolio construction to form a second order cone program, for which efficient numerical solution algorithms exist.

The constant  $\sigma$  must be exogenously chosen. If the purpose of the modified risk model is to accurately correct risk estimates, then the choice of  $\sigma$  must be made carefully. If, however, the purpose of the modified risk model is merely to identify potentially under-estimated risk portfolios and make those less desirable to a portfolio construction algorithm, then the choice of  $\sigma$  is not critical; it need only be large enough to make those portfolios less desirable. Details and specific examples illustrating the effect of different choices of  $\sigma$  are given in sections 3 and 4 below.

### 3. A Simplified Numerical Example.

The use of the Alpha Factor Method is illustrated below with a specific numerical example. Assume that there are five stocks in an investment portfolio: A, B, C, D, and E. Assume that the true asset-asset covariance matrix for these five stocks is given by  $Q$  below:

$$Q = \begin{bmatrix} 0.20 & 0.065 & 0.015 & -0.055 & -0.17 \\ 0.065 & 0.045 & 0.0075 & -0.0175 & -0.055 \\ 0.015 & 0.0075 & 0.045 & 0.0075 & 0.015 \\ -0.055 & -0.0175 & 0.0075 & 0.0289 & 0.065 \\ -0.17 & -0.055 & 0.015 & 0.065 & 0.1936 \end{bmatrix} \quad (7)$$

The rows and columns of  $Q$  correspond to the assets A, B, C, D, and E in that order.

Assume we wish to compute the risk for a portfolio whose wealth is distributed with 0% in asset A, 23.82% in asset B, 22.47% in asset C, 13.10% in asset D, and 40.61% in asset E. This asset allocation corresponds to the vector

$$w = [0 \quad 0.2382 \quad 0.2247 \quad 0.1310 \quad 0.4061]^T. \quad (8)$$

The true variance of this portfolio is given by  $V = w^T Q w = 0.03641$ , which corresponds to a volatility of 19.08%.

Now consider a one-factor, factor risk model  $Q_{(1)}$  that models  $Q$  defined by the following factor exposure, factor-factor covariance, and specific variance matrices:

$$B = [-3 \quad -1 \quad 0 \quad 1 \quad 3]^T \quad (9)$$

$$\Sigma = [0.021] \quad (10)$$

$$\Delta^2 = \text{diag}(0.010404, 0.023409, 0.0441, 0.007396, 0.004096) \quad (11)$$

For this one-factor model, the modeled asset-asset covariance matrix is;

$$Q_{(1)} = \begin{bmatrix} 0.1994 & 0.063 & 0 & -0.063 & -0.189 \\ 0.063 & 0.04441 & 0 & -0.021 & -0.063 \\ 0 & 0 & 0.0441 & 0 & 0 \\ -0.063 & -0.021 & 0 & 0.0284 & 0.063 \\ -0.189 & -0.063 & 0 & 0.063 & 0.1931 \end{bmatrix} \quad (12)$$

While many of the elements of  $Q_{(1)}$ , including the diagonals are fairly close to those of  $Q$ , others are not. In particular, the third row and third column are identically zero except for the diagonal element. This dissimilarity occurs because the one-factor factor-risk model only approximates the true asset-asset covariance matrix.

For the one-factor risk model, the variance of the portfolio  $w$  is  $V_{(1)} = w^T Q_{(1)} w = 0.03028$ , which corresponds to a volatility of 17.40%. This volatility value underestimates the true value of 19.08%.

When we compute  $B^T w$ , we obtain the vector [1.111], so  $w$  does not fully lie in the null space of  $B^T$ . Nevertheless, we can use the Alpha Factor Method to adjust the factor risk model for this particular portfolio where the magnitude of the adjustment is proportional to the extent to which  $w$  does lie in the null space of  $B^T$ . We use  $\sigma = 0.15$ . The solution for the new factor exposure to this problem for this particular  $w$  is

$$f = [0.052404 \quad 0.092362 \quad 0.070659 \quad 0.02371 \quad 0.075288]^T \quad (13)$$

The magnitude of  $f$  in (13) has already been scaled by  $\sigma$ . With this new factor, the asset-asset covariance matrix becomes<sup>1</sup>

$$Q_{(2)} = \begin{bmatrix} 0.20215 & 0.06784 & 0.00370 & -0.06176 & -0.18505 \\ 0.06784 & 0.05294 & 0.00653 & -0.01881 & -0.05605 \\ 0.00370 & 0.00653 & 0.04909 & 0.00168 & 0.00532 \\ -0.06176 & -0.01881 & 0.00168 & 0.02896 & 0.06479 \\ -0.18505 & -0.05605 & 0.00532 & 0.06479 & 0.19876 \end{bmatrix} \quad (14)$$

and the variance of the portfolio  $w$  is  $V = w^T Q_{(2)} w = 0.0354$ , which corresponds to a volatility of 18.82% which is much closer to the true value of 19.08%. Hence, even for portfolios that do not lie fully in the null space of  $B^T$ , the Alpha Factor Method can be used to improve underestimated risk estimates.

## 4. Empirical Tests.

### 4a. Backtests 1 & 2: Long/Short Strategies

Differences between the original model risk estimates and the Alpha Factor Method estimates were evaluated using two sets of backtests using Axioma's proprietary risk model for the US market. For each backtest, portfolios were constructed using a long-short, dollar-neutral, maximize-return objective portfolio management strategy over a period of ten years with monthly rebalancings. In each portfolio generated by the monthly rebalancings, there were between 900 and 1500 equities chosen from an asset universe of approximately 1800 equities. The actual number of equities varied from month to month depending on the events of the preceding month, such as mergers, IPOs, bankruptcies, etc. In addition to the risk and the dollar-neutrality constraints, asset bounds of plus or minus 5% were imposed and the long and short holdings in each

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<sup>1</sup> Although the corrected variance and volatility are uniquely defined by the Alpha Factor Method,  $Q_{(2)}$  is not.  $Q_{(2)}$  in equation (14) was computed using  $Q_{(2)} = Q_{(1)} + f f^T$  as shown in eqn. (4).

industry were restricted to be no more than 25%. Total long holdings were limited to 100% and (roundtrip) turnover for each monthly rebalancing was limited to 150%. In the first set of backtests, the goal was to limit risk to 3 percent. In the second set of backtests, the goal was to limit the risk to 6 percent.

For each risk limit, three separate backtests were performed. In the first backtest, Test A, the original risk model was used with the full risk budget (i.e., setting the risk limits to 3% and 6%, respectively). At the end of the 10 year backtest with monthly rebalancings, the realized portfolio return, risk, and Sharpe ratio were computed. In the second backtest, Test B, the original risk model was used with a reduced risk budget so that the realized risk during the backtest approximately matched the full risk budget. In the third backtest, Test C, the Alpha Factor Method together with the full risk budget was used. Prior to performing Test C, additional backtesting was performed to estimate a good value for the volatility of the unspecified factor. For these tests, we used  $\sigma = 40\%$ .

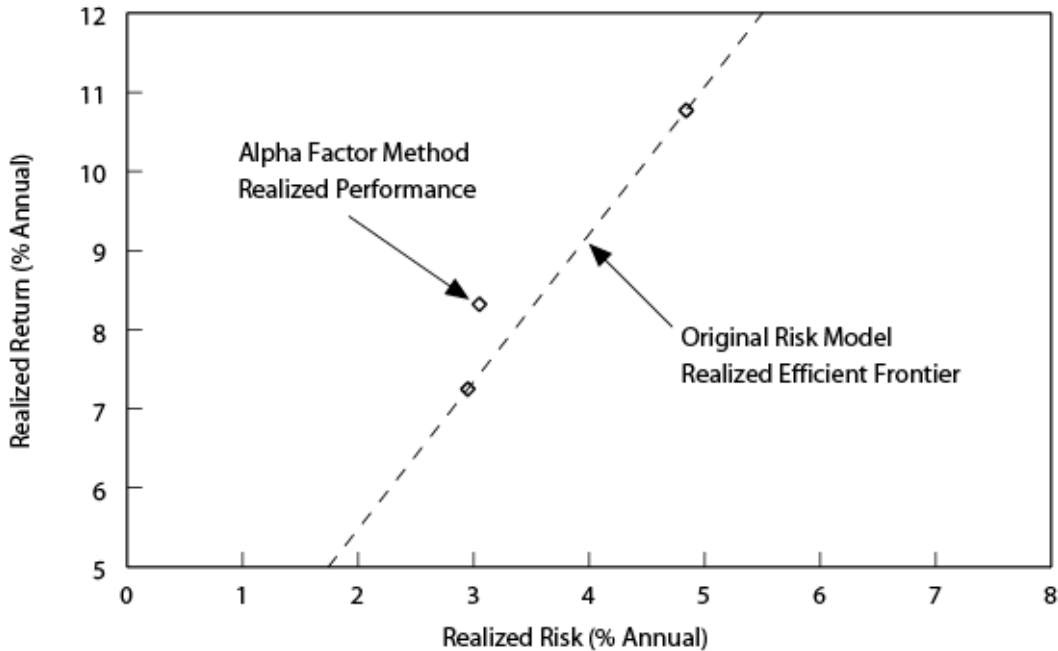
The results for the first set of backtests, Tests 1A, 1B, and 1C, in which the goal was to limit risk to 3 percent, are shown in Table 1. For Test 1A, the original risk model with the full risk target, the realized annual return was 10.77% and the realized annual risk was 4.84%, producing a Sharpe ratio of 2.23. As discussed, for these optimized portfolios, the original risk model underestimated realized risk by about 40% (4.84% versus 3%).

Back Test	Maximum Estimated Risk	Modified Risk Model?	Annual Realized Return	Annual Realized Risk	Realized Sharpe Ratio
1A	3%	No	10.77%	4.84%	2.23
1B	1.8%	No	7.25%	2.95%	2.46
1C	3%	Yes	8.32%	3.05%	2.73

**Table 1.** The first set of backtests with a 3% risk target and a long/short strategy.

Test 1B illustrates one possible approach for compensating for risk underestimation. If experience indicates that a factor risk model underestimates realized risk by, say, 40%, one can simply adjust the maximum allowable estimated risk by the same amount in the hope that the realized risk would better match the original risk target. Test 1B shows the results when the maximum estimated risk is set to 1.8%. In this case, the realized risk becomes 2.95%, which is close to the original risk target of 3%. For this backtest, the realized annual return was 7.25%, so that the resulting Sharpe ratio was 2.46, which is 10% higher than the Sharpe ratio of Test 1A.

Test 1C incorporated the Alpha Factor Method and a risk target of 3% instead of attempting to match realized and expected risk by adjusting the maximum estimated risk target downward to 1.8%. For this backtest, the annual return was 8.32% and the annual realized risk was 3.05%, a far more accurate risk result than Test 1A. The Sharpe ratio for Test 1C was 2.73, a significant improvement over both Test 1A and Test 1B. For this set of backtests, the Alpha Factor Method improved accuracy and performance when compared with either the original risk model or an approach that reduced the estimated risk budget.



**Figure 2.** The realized performance for Tests 1A, 1B, and 1C.

In Figure 2, the realized performance of the test is plotted in the efficient frontier plane, with risk on the x axis and return on the y axis. The first two tests, 1A and 1B, show the realized performance of the original risk model. A line drawn between these two points serves as a first order approximation to the realized efficient frontier of the original risk model. The third test, Test 1C, generates performance that lies above the efficient frontier of the original risk model. We have not simply moved along the original efficient frontier and thereby inadvertently changed the Sharpe ratio. We have, in fact, achieved an improved performance that lies above the original efficient frontier.

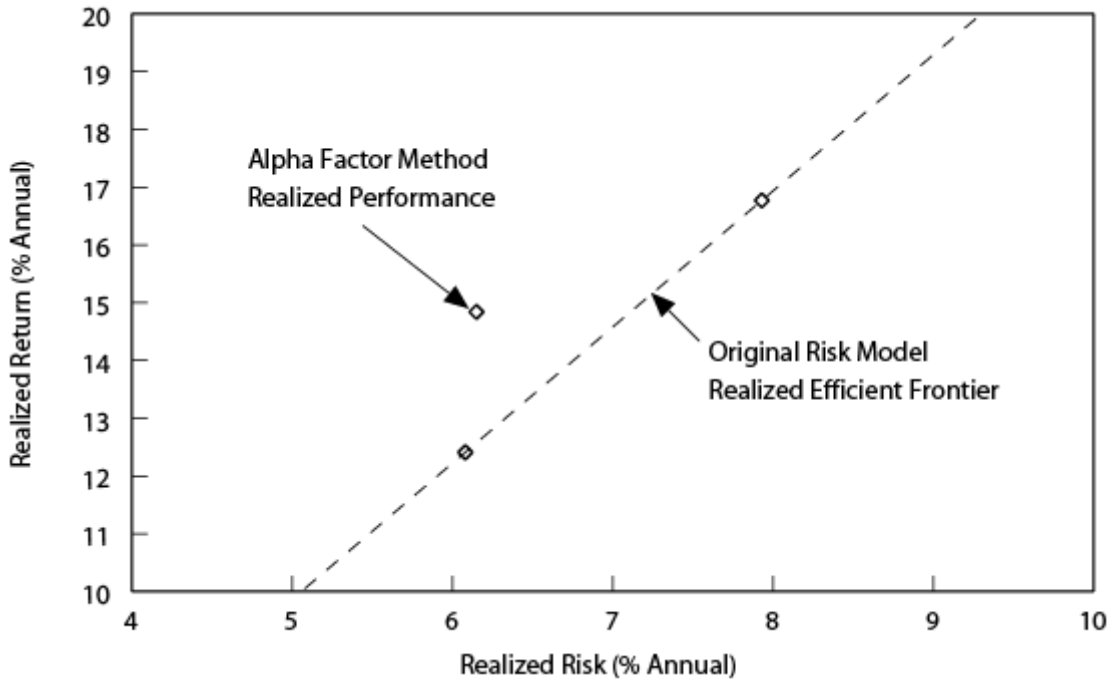
The results of a second set of backtests are shown in Table 2. In this set of backtests, Tests 2A, 2B, and 2C, the estimated risk target was 6%. Test 2A used the original factor risk model with the full risk budget. Test 2B used the original risk model with a reduced risk budget so as to make the realized risk match the 6% risk target. Test 2C used the Alpha Factor Method with the full risk budget.

In this set of backtests, the reduced risk target, Test 2B, performed worse than the original backtest, Test 2A, with the Sharpe ratio reduced from 2.11 to 2.04. However, the Sharpe ratio for the Alpha Factor Method in Test 2C was significantly higher than either of the other two backtests. In addition, the prediction of realized risk was more accurate.

<b>Back Test</b>	<b>Maximum Estimated Risk</b>	<b>Modified Risk Model?</b>	<b>Annual Realized Return</b>	<b>Annual Realized Risk</b>	<b>Realized Sharpe Ratio</b>
2A	6%	No	16.77%	7.93%	2.11
2B	4%	No	12.41%	6.08%	2.04
2C	6%	Yes	14.84%	6.15%	2.41

**Table 2.** The second set of backtests with a 6% risk budget and a long/short strategy.

The performance of Tests 2A, 2B, and 2C and their relation to the efficient frontier of the original risk model is shown in Figure 3 below. As before, the Alpha Factor Method resulted in performance that is substantially above the original efficient frontier.



**Figure 3.** The realized performance for Tests 2A, 2B, and 2C.

Tables 3 and 4 give a sense for the differences in the portfolios that occurred in backtests 1A and 1C. Table 3 shows the top three and bottom two holdings in the portfolios of Tests 1A and 1C after the first monthly rebalancing. In Test 1A with the original risk model, 1.84% of the portfolio is held in shares of ticker PEG and -3.77% is held in shares of AIV. These are the largest positive and largest negative holdings with a positive holding representing a long position in the stock and a negative holding representing a short position in the stock. For Test 1C with the adjusted risk model, the portfolio has a top positive holding of 0.51% in ticker VLO and the top negative holding is -0.70% in AIV.

Test 1A		Test 1C	
Ticker	Holdings	Ticker	Holdings
PEG	1.84%	VLO	0.51%
BXP	1.39%	MOD	0.48%
CEI	1.13%	GVA	0.46%
...		...	
CNA	-1.26%	XRX	-0.63%
AIV	-3.77%	AIV	-0.70%

**Table 3.** The top three and bottom two holdings in Tests 1A (left) and 1C (right).

Ticker	Holdings For Test 1A	Holdings For Test 1C
PEG	1.84%	0.39%
BXP	1.39%	0.25%
CEI	1.13%	0.32%
ED	1.10%	0.32%
FCE/A	0.95%	0.37%
TXU	0.87%	0.002%
...	...	...
HCP	-1.00%	-0.35%
AYE	-1.01%	-0.25%
NYB	-1.03%	-0.58%
CNA	-1.26%	-0.51%
AIV	-3.77%	-0.70%

**Table 4.** A comparison of the holdings for common tickers in Tests 1A and 1C.

In the portfolio for the adjusted risk model, there are non-zero holdings in tickers PEG, BXP, CEI, and CNA which are not listed. Table 4 shows the weights of the top six and bottom five holdings for the portfolio after the first month's rebalancing for Test 1A. These holdings include portfolio weights for tickers PEG, BXP, CEI, ED, FCE/A, TXU, HCP, AYE, NYB, CNA AND AIV. The right-hand column of Table 4 also shows the holdings of these same tickers in Test 1C. All holdings for these particular tickers have decreased in absolute value.

#### 4b. Backtests 3: Long Only Strategies

A third set of backtests was performed using a long-only strategy. For each backtest, portfolios were constructed using a long-only, fully-invested, maximize-return objective portfolio management strategy over a period of ten years with monthly rebalancings. In each portfolio generated by the monthly rebalancing, there were between 90 and 850 equities chosen from an asset universe of approximately 1000 equities. The only constraints were the active risk constraint and the long-only, fully-invested constraint. The goal was to limit active risk versus a benchmark to 3 percent. The benchmark was chosen as the market portfolio of the top 1000 stocks by market capitalization. For this test,  $\sigma = 20\%$  was used. Two tests were performed: Test 3A, which used the original risk model and the full risk budget; and Test 3C, which used the adjusted risk model with the full risk budget.

The results of this set of backtests are shown in Table 5. For Test 3A, the realized annual risk was 3.33%, while for Test 3C, it was 2.92%. As with the previous tests, these results indicate that the Alpha Factor Method improves the accuracy of the original risk model. The Sharpe ratio of Test 3C was substantially higher than that produced by Test 3A, increasing in size from 2.11 to 2.81. Hence, by both measures – risk estimation accuracy and realized Sharpe ratio – inclusion of the Alpha Factor Method improves both the risk estimation accuracy and realized Sharpe ratio when compared against application of the original risk model alone.

Back Test	Maximum Estimated Risk	Modified Risk Model?	Annual Realized Return	Annual Realized Risk	Realized Sharpe Ratio
3A	3%	No	7.92%	3.33%	2.11
3C	3%	Yes	8.21%	2.92%	2.81

**Table 5.** The third set of backtests with a 3% risk budget and a long-only strategy.

#### 4c. Additional Backtests – Different Return Estimation Models

In order to determine to what extent the results were dependent upon the particular choice of estimating returns, an additional series of backtests were performed with several different methods for estimating returns.

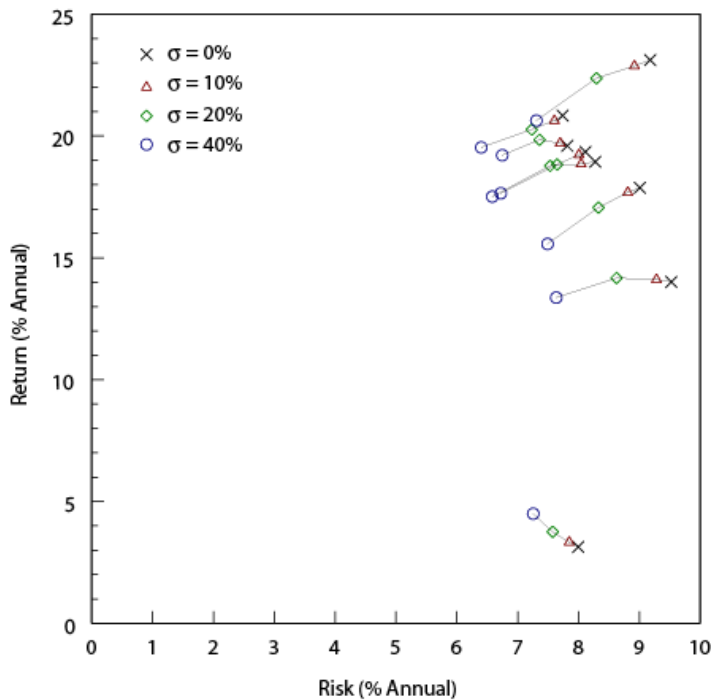
Eight different return models were constructed by regressing a different set of fundamental factors over the 1995 – 2005 time period. Each of these eight unique fundamental returns was used in a backtest to predict returns for each monthly rebalancing. In addition to the eight fundamental return models, ten return models were constructed which generated random returns for the assets over the same time window but with an information coefficient of 5%.

As before, two different portfolio construction strategies were employed: a long/short strategy and a long-only strategy. The sensitivity of the results to the choice of  $\sigma$  was investigated by running each of the backtests with three different values of  $\sigma$ : 10%, 20%, and 40%. We also ran the backtests without using the Alpha Factor Method. These results are denoted by  $\sigma = 0\%$ .

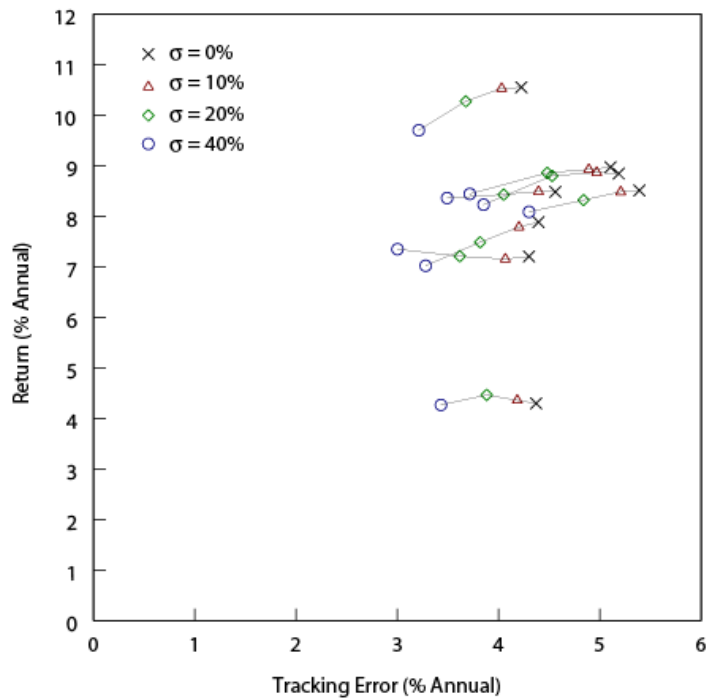
The results for each case are plotted in Figs. 4 - 7 below. Each Figure shows the average monthly return versus risk (or tracking error) for the backtests. The first two graphs show the results for the eight fundamental return models, the long/short strategy in Fig. 4 and the long-only strategy in Fig. 5. Figs. 6 and 7 show the results for the ten random return models, the long/short strategy in Fig. 6 and the long-only strategy in Fig. 7. For each separate model, there are four points plotted: a black X corresponding to  $\sigma = 0\%$  which represents the original risk model; a red triangle corresponding to  $\sigma = 10\%$ ; a green diamond corresponding to  $\sigma = 20\%$ ; and a blue circle corresponding to  $\sigma = 40\%$ . The four points for each model are connected by a thin black line. Moving from the

black X to the blue circle shows the effect of increasing the magnitude of the Alpha Factor Method compensation.

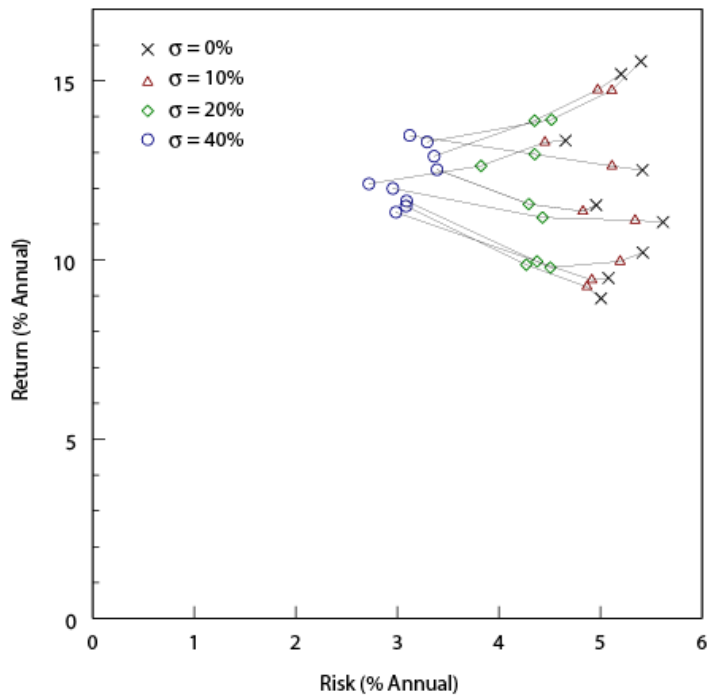
In all four graphs, the principle effect of the Alpha Factor Method is to move realized performance to the left in the risk/return plane. This corresponds to reduced realized risk (or tracking error). This is exactly what is meant to happen with the Alpha Factor Method: underestimated risk portfolios are not selected, so the portfolios that are selected have realized risks that are lower and more accurate than they would be without use of the Alpha Factor Method. In many cases, the resulting return stays the same or even (in the case of many of the random, long/short strategies) increases. In other cases, the return decreases slightly. The graph does not show the slope of the original efficient frontier, so it is not possible to say whether or not the realized performance of the Alpha Factor Method lies above that curve. It has been our experience, however, that the realized performance is above the original efficient frontier.



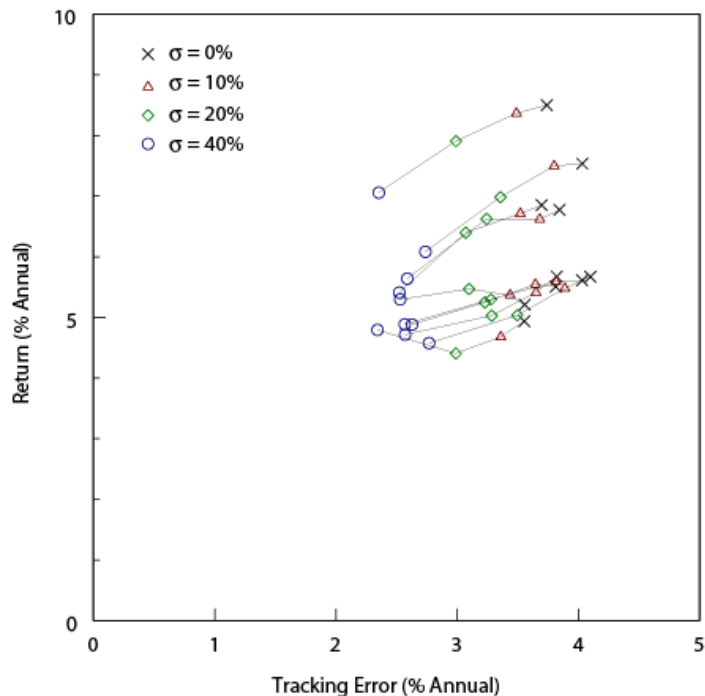
**Figure 4.** The Alpha Factor Method backtest results for eight different fundamental return models with a long/short strategy.



**Figure 5.** The Alpha Factor Method backtest results for eight different fundamental return models with a long-only strategy.



**Figure 6.** The Alpha Factor Method backtest results for ten different random return models with a long/short strategy.



**Figure 7.** The Alpha Factor Method backtest results for ten different random return models with a long-only strategy.

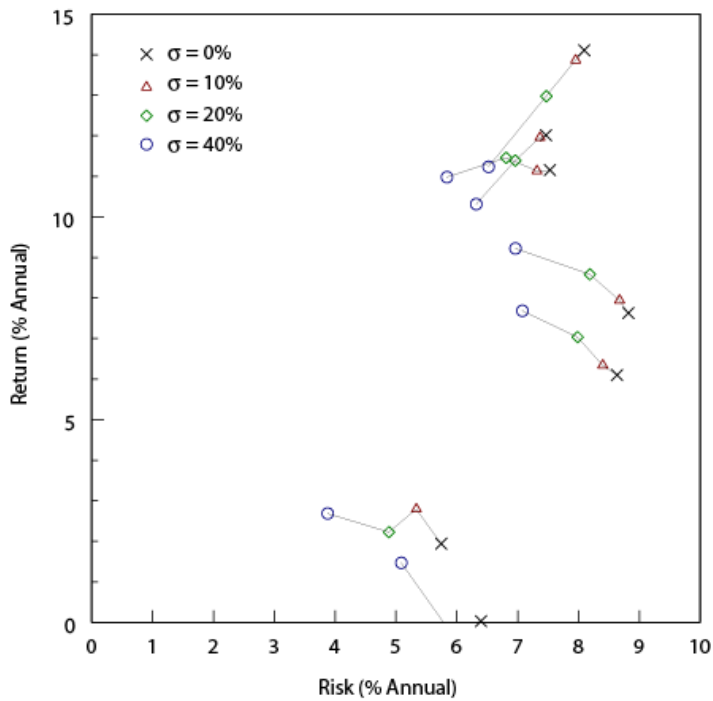
#### 4d. Different Return Estimation Models with a Turnover Constraint

An additional set of backtests was performed using the eight fundamental return models of the previous section. As in the previous section, long/short and long-only strategies were tested. In these backtests, however, the maximum monthly turnover was set to 20%.

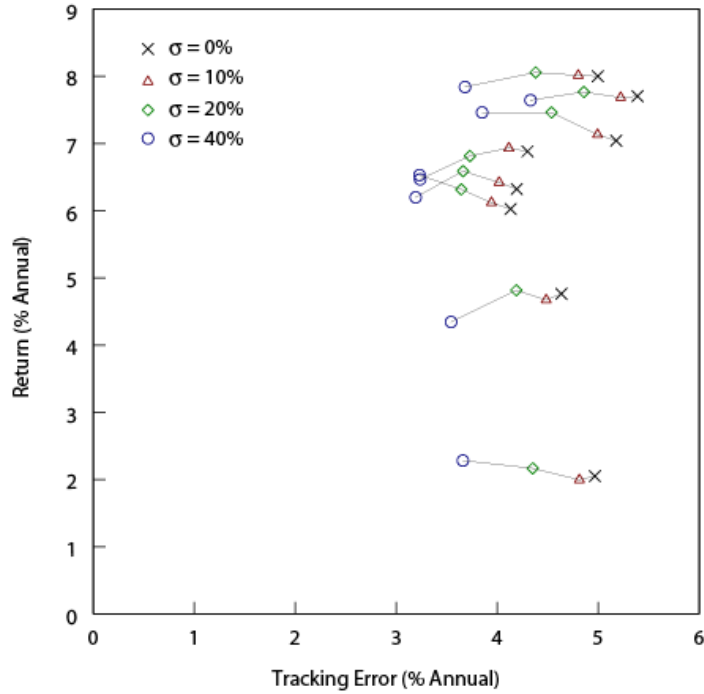
Figures 8 and 9 show the results for these tests, using the same plotting conventions as in the previous section. The long/short strategy is shown in Fig. 8 and the long-only strategy is shown in Fig. 9. The results for each model are connected by a line, and the four different values of  $\sigma$  (0, 10, 20, and 40%) are shown by different points.

As in the previous section, the principle effect of the Alpha Factor Method is to move realized performance to the left in the risk/return plane. This is caused by more

accurate risk estimation: the portfolios with underestimated risk are removed from consideration by the Alpha Factor Method, with the resulting realized risk or tracking error being lower. The effect on realized return differs for different return models. In some cases, the realized return increases as  $\sigma$  increases. In other cases, realized return decreases slightly with increases in  $\sigma$ .



**Figure 8.** The Alpha Factor Method backtest results for eight different fundamental return models with a long/short strategy and the maximum monthly turnover set to 20%.



**Figure 9.** The Alpha Factor Method backtest results for eight different fundamental return models with a long-only strategy and maximum monthly turnover set to 20%.

## 5. Summary and Conclusions.

A new patent-pending method termed the Alpha Factor Method has been described. This method provides an improved method for utilizing any factor risk model. The Alpha Factor Method can be used to either improve risk estimation or to penalize portfolios whose risk is suspected of being underestimated so that they are less desirable during portfolio optimization. Extensive numerical results show that the Alpha Factor Method increases accuracy and improves portfolio performance.

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