Portfolio Construction Research by

Using Axioma's Alpha Factor Method To Correct the Misalignment of Alpha Model and Risk Model Factors

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1. Introduction

Factor models are used to predict and explain the expected return and risk of portfolios. Typically, the number of factors used in an alpha or risk model is much lower than the number of assets in the portfolio, or in the investable universe. This short note studies the consequence of this "dimensionality gap" and discusses a methodology, Axioma's Alpha Factor™ method[1], that limits the potential impact of the difference between the asset space and the factor space. Real world backtest results show improved portfolio performance when this methodology is used.

2. Most Portfolios Lie in the Factor Null Space

The properties of factor models derived from linear algebra, and, in particular, the decomposition of the vector space of assets covered by the model into the image space (the space spanned by the factors) and the null space (the space that cannot be described by the factors), has been the focus of several recent research efforts. The central objective in these studies has been distinguishing information from noise in the null space. Miller (2006) proposed a hybrid factor risk model in order to capture the risk contribution of factors not formally defined by the risk model – that is, from factors lying in the null space. Menchero and Mitra (2008) performed additional analysis of

[1] Axioma's Alpha Factor™ method is patent-pending.

Miller's hybrid models. Lee and Stefek (2008) demonstrated that when the factors used in an alpha model are different than those used in a risk model (the difference, of course, lies in the null space), optimizers can create portfolios that may underperform.

The potential impact of the factor model null space on portfolio construction is stronger than is generally realized. To illustrate this, Axioma obtained real world, US backtest data including historical alphas and portfolio construction parameters (e.g., asset bounds, tracking errors, etc.) from thirteen portfolio management teams and performed backtests using all the data sets. The backtests comprised a total of 1625 monthly rebalancings ranging from January 1995 to July 2008. All strategies maximized expected return with a fixed tracking error constraint. The benchmarks involved included the S & P 500, and the Russell 1000, 3000, and 2000 Value. At each rebalancing, the portfolio of optimized active holdings was computed, W, and its projection into the null space was computed using Axioma's US risk model, W_N[2]. The ratio of the magnitudes[3] of W_N to W was then computed. The aggregate distribution is shown in Figure 1.

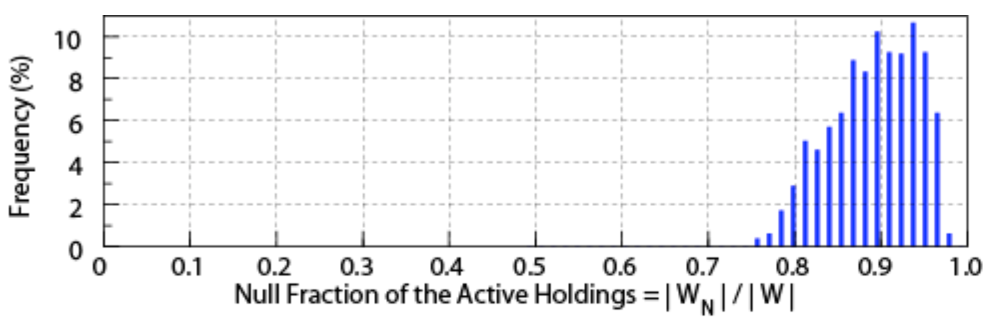

Figure 1. The distribution of the fraction of the active portfolio in the null space for a set of 1625 real world portfolio rebalancings.

For these real world portfolio construction strategies, ***over 75% of all, rebalanced active holdings lie in the null space***.

In Figure 2, we show the projection of the alphas (expected returns) used for the backtest into the null space.

[2] Formally, W is a column vector of active holdings (managed holding minus the benchmark); W_N is the projection of W into the null space defined by the risk model's exposure matrix.

[3] For simplicity, we define the magnitude as the ℓ^2-norm $|W| = \sqrt{\sum_i W_i^2}$. The magnitude of the projection is always less than or equal to the magnitude of the original vector.

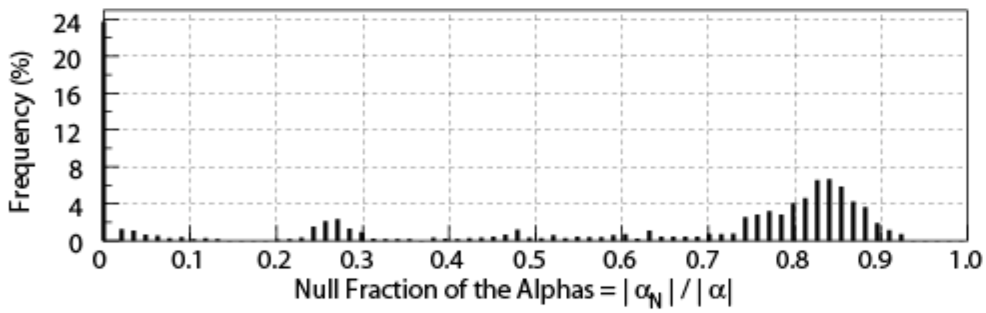

Figure 2. The distribution of the fraction of the expected returns (alpha) in the null space for the same backtests shown in Fig. 1. For legibility, zero has been offset on the horizontal axis. Also note the change in vertical scale.

Almost 24% of the alphas lie completely in the image space, not the null space. From conversations with the portfolio managers who created several of the historical sets of alphas, we know that 3 of the 13 used Axioma's risk factors as drivers for generating alphas. These 3 sets account for the vast majority of the alphas with no exposure to the null set.

However, almost 50% of the alphas have at least 75% of their magnitude in the null space, the same range in which all of the optimized portfolios occur. This indicates that the potential for misalignment of the alpha and risk model factors can vary dramatically depending on the alpha generation process used.

There are at least three, possible, interrelated explanations of why optimized portfolios have such a large projection into the null space:

(1) **Opportunity-driven**. Because the "dimensionality gap" between the vector space of portfolio holdings and the vector space of factors is so large, most portfolios, even random portfolios, will have large projections into the null space. Axioma's US risk model has only 76 factors and the minimum universe size in the backtests is 500. The null space therefore has at least 424 dimensions and even more for the larger benchmarks.

(2) **Risk-driven**. Since the factor risk associated with any of the null space dimensions is zero, portfolios with a large projection into the null space have low factor risk and typically have low total risk. Portfolio rebalancings that are driven by tight risk constraints are therefore likely to choose portfolios lying in the null space.

(3) **Alpha-driven**. Since the alphas often have large projections into the null space, portfolio rebalancings that maximize expected return are likely to choose portfolios with similar projections.

Portfolio managers may well ask (and test) whether or not there is a performance difference between those alphas that lie in the null space and those that do not. Do the alphas in the null space represent valuable information not embedded in the risk model that the manager wishes to take bets on, or do they generally have so much noise in their predictions that the manager would rather limit the bets taken on them?

Axioma's Alpha Factor™ method can be used to reduce the optimized portfolio's exposure to the null space. Studies have shown that the Alpha Factor method often leads to better realized performance, generally in terms of reduced realized volatility (see, for example, Renshaw et al., 2006), by limiting a portfolio's exposure to potentially noisy information and risk predictions.

In Figure 3, we show the same backtest results using Axioma's Alpha Factor™ method. The portfolios generated using Axioma's Alpha Factor method have significantly less exposure to the null space.

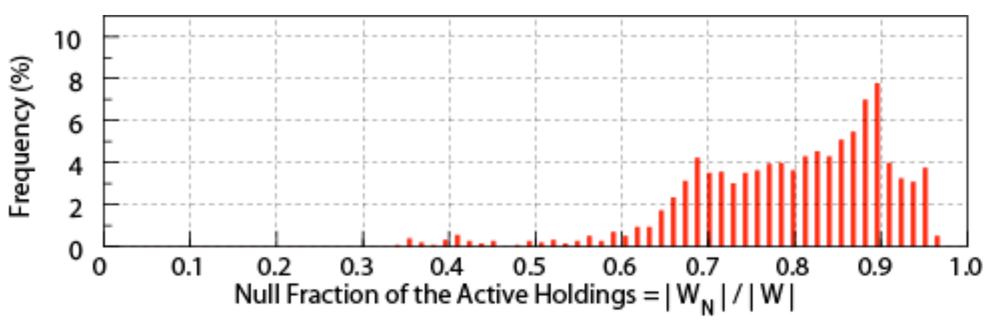

Figure 3. The distribution of the fraction of the active portfolio in the null space using Axioma's Alpha Factor method.

3. Improved Realized Performance Using the Alpha Factor Method

Here we document the improved, realized portfolio performance generated using Axioma's Alpha Factor method. We compare three different set of backtests:

(a) **No Alpha Factor**. These are the portfolios for the original 13 sets of real world backtests and strategies without the Alpha Factor method whose rebalanced portfolios are shown in Fig 1.

(b) **Traditional Alpha Factor**. These are the portfolios using the traditional Alpha Factor method which penalizes portfolios that lie in the null space of the risk model exposure matrix, *B*. These portfolios are shown in Fig. 3.

(c) **Modified Alpha Factor**. These are the portfolios generated using the Alpha Factor method modified to penalize portfolios that lie in the null space of the union of the risk and alpha model factors. In our study, we only possess a single factor associated with the expected returns, α, the column vector of expected returns. We therefore use Axioma's Alpha Factor method using the null projection matrix associated with the matrix $\left[B \quad \alpha\right]$.[4]

Figure 5 shows the distribution of realized, monthly returns for the backtests of cases (a) and (b). The blue distribution is without the Alpha Factor method (case a). The red distribution uses the Alpha Factor method (case b).

[4] If a portfolio manager had several different alpha factors, he or she could run the modified Alpha Factor method using the modified exposure matrix of $\left[B \quad \alpha_1 \quad \ldots \quad \alpha_n\right]$ to account for all of these alpha factors. Or, if a portfolio manager or risk model provider performed a PCA analysis of the residual returns, the modified exposure matrix could also include one or more of the principal components describing specific risk.

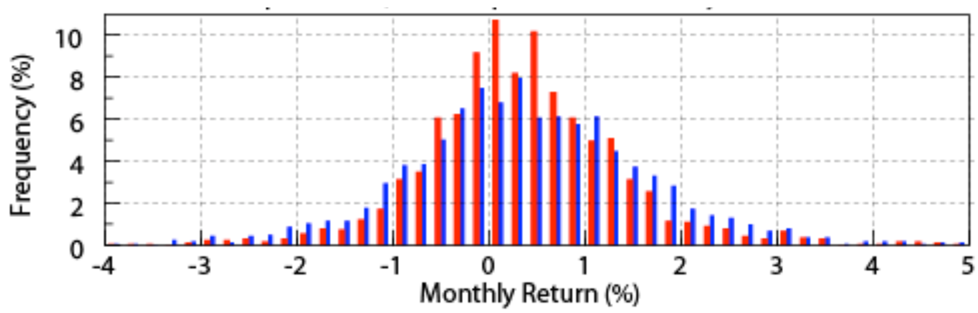

Figure 5. The distribution of realized monthly returns with (red – case b) and without (blue – case a) the Alpha Factor method for the backtests shown in Figs. 1 and 3.

Without the Alpha Factor (the blue bars), the distribution is wider than when the Alpha Factor method is present. When the Alpha Factor is present, the returns are more concentrated near the mean.

The average returns, volatilities, and information ratios associated with all three sets of backtest are shown in the table below.

Case	Average Monthly Return	Monthly Volatility	Information Ratio
(a) No Alpha Factor	0.398%	1.349%	1.021
(b) Traditional Alpha Factor	0.381%	1.116%	1.183
(c) Modified Alpha Factor	0.389%	1.114%	1.184

Table 1. Realized performance for all three cases.

The Alpha Factor produces a modest reduction in the average monthly return together with a more significant reduction in the realized volatility in both cases (b) and (c). As a result, the information ratio increases from 1.02 to 1.18. This represents a significant boost in realized portfolio performance. The difference between cases (b) and (c) is quite modest, indicating that the alpha model space does not add any appreciable information to the factor space defined by the risk model, at least for these backtests.

4. Discussion and Conclusions

The misalignment of the factors of an alpha model and a risk model and the potential impact of this misalignment on portfolio optimization has been a fashionable research topic (Lee and Stefek

2008). What seems to have been overlooked, however, is the fact that the factor null space is always large and always present even when the alpha model and risk model factors are perfectly aligned (e.g., the same). Furthermore, opportunity, risk and alpha all are likely to drive optimizers to choose portfolios in that lie in the null space. Although the results presented here are limited to only 13 sets of real world backtests, they do suggest that the impact of the null space is far more significant than the misalignment of the factor spaces, even when the misalignment is significant. In our results, over 50% of the alphas had over 75% of their magnitude in the null space; yet, the difference in the Information Ratio with and without accounting for the difference was only 0.001.

Finally, note that the improved performance shown in Table 1 represents the aggregated results of 13 different real-world backtests, each of which has its own set of alphas and portfolio construction strategies. This increases the number of rebalancings used in the study, which reduces the error on the statistics reported. We would expect the performance improvement to be even greater if each individual case where calibrated separately.

There are three main take-aways from this piece:

- The impact of the noise associated with the null space of alpha and risk model factors is larger than generally realized because in most real world situations, the majority of the portfolio lies in the null space.
- Use of the Alpha Factor can significantly reduce a portfolio's exposure to the null space. Furthermore, use of the Alpha Factor can significantly improve the portfolios realized performance.
- When alpha and risk factors are not identical, the difference of the factors lies in the null space. A modified Alpha Factor method has been shown to perform slightly better than the traditional Alpha Factor method for the data analyzed here.

5. References

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